Research group : Geometric Invariant Theory

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Chapter 1

Introduction to GIT

Geometric invariant theory arises in an attempt to construct a quotient of an algebraic variety X by an algebraic action of a linear algebraic group G. In many applications X is the parametrizing space of certain geometric objects (algebraic curves, vector bundles, etc.) and the equivalence relation on the objects is defined by a group action. The main problem here is that the quotient space X/G may not exist as an algebraic variety. The main reason to this fact is that the orbits may be non closed.

The main idea is to restrict this action to a Zariski open subset $U \subset X$ such that the quotient $U \to U/G$ exists as quasi-projective algebraic variety and U is maximal in some sense. This bring us to the question : how do we choose this U ?

The three main references for these lectures are, obviously the "bible of GIT" : Geometric invariant theory by Mumford [1], the Dolgachev's book : Lecture on invariant theory and also Newstead [2] an introduction to moduli problems and orbit spaces.

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1.1 Actions

Definition. An (left) *action* of an algebraic group G on a variety X is a morphism :

$$\sigma : G \times X \to X$$

such that

- $\sigma(g, \sigma(g', x)) = \sigma(gg', x)$,
- $\sigma(e, x) = x$.

We said that (X, σ) is a *G-variety*.

We drop the σ and write just $g.x$ for $\sigma(g, x)$.

We will denote by O_x (*resp.* St_x) the orbit (*resp.* the stabilizer) of x under the action of G.

Definition. A *G-morphism* ϕ between two varieties X and Y is a G-equivariant morphism and it is G-invariant if it is constant on orbits.

Definition. Let G be an algebraic group. A *rational representation* of G is a morphism $G \to GL_n(k)$ and the corresponding action on k^n is called a *linear action* of G on k^n.

Note that, given a G-variety X, we can define an automorphism of the k-algebra $\mathcal{O}(X)$ by sending $f : x \mapsto f(x)$ to $g^*f : x \mapsto f(g.x)$. And we have :

Lemma 1. *Let X be a G-variety and W a finite-dimensional subspace of $\mathcal{O}(X)$. Then*

- *if W is invariant then the action of G on W is given by a rationnal representation,*
- *in any case, W is contained in an invariant finite-dimensional subspace.*

Proof. Let $f_1, \cdots, f_n$ be a basis of W then

$$g^*f_i = \sum_{j=1}^{n} \rho_{ij}(g) f_j, \ \rho_{ij}(g) \in k$$

and $g \mapsto \rho_{ij}(g)$ give a rationnal representation.

For the second statement, we just need to check that $\mathrm{Span}(g^*f_1, \cdots, g^*f_n)$ for all $g \in G$ is finite-dimensional (see [2]). □

1.2 Categorical and geometric quotients

Definition. Let (X, σ) be a G-variety. A *categorical quotient* of X by G is a pair (Y, ϕ) where Y is a variety and $\phi : X \to Y$ is a G-invariant morphism such that any other G-invariant morphism $f : X \to Z$ there exists a unique morphism $\psi : Y \to Z$ such that $f = \psi \circ \phi$.

Moreover, if $\phi^{-1}(y)$ consists of a single orbit for all $y \in Y$, we call (Y, ϕ) an *orbit space*.

Proposition 1. *A categorical quotient is determined up to isomorphism.*

Example. Let $GL_n(k)$ act on $M_n(k)$ by conjugation. The pair (k^n, χ) with $\chi : M_n(k) \to k^n$ given by the characteristic polynomial is a categorical quotient.

Let prove it for $n = 2, k = \mathbb{C}$. Let

$$f : M_2(\mathbb{C}) \to Z$$

be a $GL_2(\mathbb{C})$-invariant morphism. As it is constant along orbits, one can consider the Jordan form to distinguish orbits. We obtain three types of matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \qquad \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \qquad \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$$

which are not similar. But the matrices

$$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \qquad \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

have same characteristic polynomial. If we consider

$$B_t := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} \alpha & t^2 \\ 0 & \alpha \end{pmatrix}$$

we get that $f(B_t) = f(B_1)$ for all $t \neq 0$ and hence also for $t = 0$.

One can consider the morphism $p : \mathbb{C}^2 \ni v \to C_v \in M_2(\mathbb{C})$ which associate to a vector v the compagnion matrix associated so that we can form the map

$$\psi : \mathbb{C}^2 \to Z, \qquad v \mapsto f(C_v)$$

which is morphism.

Remark.

- Note that (k^n, χ) is not an orbit space.

  In fact, for $\chi(\mathrm{Id}) = (-2, 1)$ and $\chi^{-1}((-2, 1)) = \mathrm{O}_{\mathrm{Id}} \cup \mathrm{O}_I$ where $I = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
- This construction of a 1-parameter subgroup $\mathbb{G}_m \to GL_2(\mathbb{C})$ acts on the G-variety is the main idea of stability of Mumford, we will generalize this condition later.

Definition. Let (X, σ) be a G-variety. A *good quotient* of X by G is a pair (Y, ϕ) where Y is a variety and $\phi : X \to Y$ is an affine G-invariant surjective morphism such that

- if U is open in Y, then

  $$\phi^* \mathcal{O}(U) \to \mathcal{O}(\phi^{-1}(U))$$

  is an isomorphism onto $\mathcal{O}(\phi^{-1}(U))^G$
- if W is closed, then $\phi(W)$ is closed,

- If W_1, W_2 are closed disjoint subset of X then $\phi(W_1) \cap \phi(W_2) = \emptyset$.

Moreover, if (Y, ϕ) is an orbit space, then we call it a *geometric quotient*.

Remark. The concepts of good (*resp.* geometric) quotient (Y, ϕ) are local with respect to Y in the sense that

- if U is open in Y then (U, ϕ) is a good (*resp.* geometric) quotient for $\phi^{-1}(U)$,
- if $\{U_i\}$ is an open covering of Y such that (U_i, ϕ) is a good (*resp.* geometric) quotient of $\phi^{-1}(U_i)$ then (Y, ϕ) is a good (*resp.* geometric) quotient of X.

Proposition 2. *A good quotient is a categorical quotient.*

Example. For $M_2(\mathbb{C})$ the $GL_2(\mathbb{C})$-variety, the categorical quotient is not a good quotient since det is not closed.

1.3 Affine quotients

We start by look at the case where X is affine.

Given an affine G-variety X we can expect that there exists a categorical quotient (Y, ϕ) with Y affine. Notice that for a categorical quotient (Y, ϕ), any G-invariant morphism $f : X \to k$ factors through ϕ. In algebraic terms this means that

$$\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$$

is an isomorphism onto the G-invariant $\mathcal{O}(X)^G$. Hence, Y is affine if, and only if, $\mathcal{O}(X)^G$ is finitely generated.

This is a version of Hilbert's fourteenth problem and Nagata gave a counterexample and a sufficient condition on G. To state this theorem, we need two definitions.

Definition. An algebraic group G is *geometrically reductive* if given a finite-dimensional rationnal representation V of G and an invariant vector $v \in V$ there exists an G-invariant homogeneous polynomial function $f : V \to k$ such that $f(v) = 1$.

Proposition 3. *Assume* $char(k) = 0$.

- *Every finite group is geometrically reductive,*
- *$SL_n(k)$, $GL_n(k)$ are geometrically reductive.*

Definition. Let G be an algebraic group and R be a k-algebra. A *rationnal action* of G on R is a map $R \times G \to R$ such that

- $(gg').f = g'.(g.f)$ and $e.f = f$,

- $f \to g.f$ is a k-algebra automorphism of R,
- every element of R is contained in a finite-dimensional G-invariant subspace on which G act by rationnal representation.

Theorem 1. *(Nagata) Let G be a geometrically reductive group acting rationnally on a finitely generated k-algebra then R^G is also finitely generated.*

The proof is too long we won't do it here, see [2] for example.

Remark. In characteristic 0, the usual notion of reductive group (with trivial unipotent radical) is equivalent to geometrically reductivity, hence we drop the term geometrically.

Theorem 2. *Let X be an affine G-variety with G reductive. There exists a good quotient (Y, ϕ) with Y affine.*

By Nagata theorem, we know that $\mathcal{O}(X)^G$ is finitely generated so that $\mathrm{Spm}(\mathcal{O}(X)^G)$ is an affine variety, we take Y as this variety.

We need :

Lemma 2. *Let X be a G-variety with G reductive and Z_i, Z_2 closed subsets of X, then there exists $f \in \mathcal{O}(X)^G$ such that $f(Z_1) = 0$ and $f(Z_2) = 1$.*

Proof. Since Z_1 and Z_2 are disjoint closed subset, the sum of the ideals defining Z_1 and Z_2 is the whole ring $\mathcal{O}(X)$, hence one can find $\alpha \in I(Z_1)$ and $\beta \in I(Z_2)$ such that $1 = \alpha + \beta$. If we consider α we have the propertie $\alpha(Z_1) = 0$ and $\alpha(Z_2) = 1$. By lemma 1 (that said that for an invariant closed subset W, the action of G restricted to W is given by a rationnal representation), we know that the subspace $W \subset \mathcal{O}(X)$ spanned by $g^*\alpha$, $g \in G$ is finite dimensional. Let $\phi_1, \cdots, \phi_n$ be a basis of W and consider the map : $X \to \mathbb{A}^n$ defined by these functions. Then $f(Z_1) = (0, \cdots, 0)$ and $f(Z_2) = (1, \cdots, 1)$. G acts by a rationnal representation of $\mathbb{A}^n$. By definition of geometrically reductive group, we can find a G-invariant homogeneous polynomial $F \in k[X_1, \cdots, X_n]$ such that $F(1, \cdots, 1) \neq 0$ then $f^*F = F(\phi_1, \cdots, \phi_n)$ satisfies the assertion. ☐

We start the proof of the theorem.

Proof. Suppose ϕ is not G-invariant then there exists $g \in G$ and $x \in X$ such that $\phi(g.x) \neq \phi(x)$. Since Y is affine, there exists $f \in \mathcal{O}(Y)$ such that $f(\phi(g.x)) \neq f(\phi(x))$ which contradicts that $\phi^*f \in \mathcal{O}(X)^G$.

We now prove the first condition. Since localisation commutes to taking invariant, one can take Y_f for some $f \in \mathcal{O}(X)^G$ as a basis of open sets and we get $(\mathcal{O}(X)^G)_f = (\mathcal{O}(X)_f)^G$.

For the last condition, by the previous lemma, we take $f \in \mathcal{O}(X)^G$ such that $f(W_1) = 0$ and $f(W_2) = 1$. Thus by the previous point, seeing f in $\mathcal{O}(Y)$, $f(\phi(W_1)) = 0$ and $f(\phi(W_2)) = 1$ hence $\overline{\phi(W_1))} \cap \overline{\phi(W_2))} = \varnothing$.

The second point, consider W closed in X and $y \in \overline{\phi(W)} - \phi(W)$. We apply the previous point to $W_1 = W$ and $W_2 = \phi^{-1}(y)$ and get a contradiction. □

Proposition 4. *Let X be a G-variety and (Y, ϕ) be a good quotient. Then*

- $\phi(x_1) = \phi(x_2) \Leftrightarrow \overline{\mathrm{O}_{x_1}} \cap \overline{\mathrm{O}_{x_2}} \neq \varnothing$,
- *if the G-action on X is closed, i.e. all the orbits are closed, then (Y, ϕ) is a geometric quotient.*

Definition. Let X be an affine G-variety. A point $x \in X$ is called *stable* if its orbit is closed and of the same dimension of G, We denote by X^s the set of stable points of X.

Proposition 5. *Let X be an affine G-variety and (Y, ϕ) a good quotient then there exists Y' a subset of Y such that $\phi^{-1}(Y') = X^s$ and (Y', ϕ) is an orbit space for X^s.*

Proof. First, a remark : $dim\ \mathrm{O}_x = dim\, G - dim\ \mathrm{St}_x$ and $x \rightarrow dim\ \mathrm{St}_x$ is an upper semi-continuous fonction of x. So that $X^{max} := \{x \in X |\, dim\, \mathrm{O}_x \geqslant n\}$ is an open set.

Consider $Y' = Y - \phi(X - X^{max})$ which is open by the previous remark and the theorem 2 ((Y, ϕ) is a good quotient and if Z is closed then $\phi(Z)$ is too).
We will show that $\phi^{-1}(Y') = X^s$. Let $x \in X'$, then the set O_x and $X - X^{max}$ are disjoint closed subset of X so that $\phi(x)$ is in Y', thus we have $X' \subset \phi^{-1}(Y')$. For the other inclusion, let $x \notin X'$ then either $x \in X - X^{max}$ or O_x is open. If $x \in X - X^{max}$ then $\phi(x) \notin Y'$. If O_x is open, let $y \in \overline{\mathrm{O}_x} - \mathrm{O}_x$ then $dim\ \mathrm{O}_y < \mathrm{O}_x$ by the remark hence $y \notin X^{max}$ and $\phi(y) = \phi(x) \notin Y'$. We conclude that in both cases, $x \notin \phi^{-1}(Y')$.

By definition of X', the action of G restricted to X' is closed and (Y', ϕ) is an orbit space. □

1.4 Projective quotients

The results in affine case does not apply in the projective one. One way to construct a quotient for a group action on a projective variety is to consider open G-invariant affine covering of X and glue the quotients together. But in general it is not possible to cover X in this way.

However, it is necessary to consider affine open subsets of X of the form X_f for f an homogeneous polynomial in $k[X_0, \cdots, X_n]$ and look for the G-invariants but G does not determine an action on this polynomial ring. This leads to this definition :

Definition. A *linearisation* of an action of an algebraic group G on a projective variety X in $\mathbb{P}^n$ is a *linear action* of G on k^{n+1} which induces the action of G on X. A linear action is an action with a linearisation of it.

Remark.

- The definition in then justify by the fact that a linear action of G on X determined a G-action on the polynomial ring $k[X_0, \cdots, X_n]$.

- A problem that we have with this definition is that given a projective G-variety X, a linearisation of the action depends obviously of the action of G but also on the embedding of X in $\mathbb{P}^n$.

We will first keep this definition even if it depends on an embedding. In a second part, we will introduce a more general notion of linearisation to avoid this issue.

Definition. Let X be a projective G-variety in $\mathbb{P}_n$ with a linearisation of the action of G. A point $x \in X$ is called

- *semi-stable* if there exists an invariant homogeneous polynomial f of degree at least 1 such that $f(x) \neq 0$,
- *stable* if $dim\ \mathrm{O}_x = dim\, G$ and there exists an invariant homogeneous polynomial f of degree 1 such that $f(x) \neq 0$ and the action of G on X_f is closed.

Remark. The definition of stable correspond of Mumford's definition of properly stable.

We denote by X^{ss} (*resp.* X^s) the set of semi-stable (*resp.* stable) points of X.

Lemma 3. *Both sets X^{ss} and X^s are open in X.*

Theorem 3. *Let X be a projective variety in $\mathbb{P}^n$. Then for any linear action of a reductive group G on X*

- *there exists a good quotient (Y, ϕ) of X^{ss} by G and Y is projective,*
- *there exists an open subset Y^s of Y such that $\phi^{-1}(Y^s) = X^s$ and (Y, ϕ) is a geometric quotient of X^s,*
- *for all $x_1, x_2 \in X^{ss}$,*

$$\phi(x_1) = \phi(x_2) \Leftrightarrow \overline{\mathrm{O}_{x_1}} \cap \overline{\mathrm{O}_{x_2}} \cap X^{ss} \neq \varnothing$$

- *for x semi-stable point,*

$$x \in X^s \Leftrightarrow dim\ \mathrm{O}_x = dim\, G, \text{ and } \mathrm{O}_x \text{ is closed in } X^{ss}$$

Remark. With the Mumford's definition of stability, we must replace $dim\ \mathrm{O}_x = dim\, G$, by $dim\ \mathrm{O}_x$ is contant in a neighbourhood of x.

We won't prove this theorem, instead we want a definition that takes in count the dependance of the embedding of X in $\mathbb{P}^n$. Before to go in this way, we show an example.

Example. Consider the following action of $\mathbb{G}_m$ on $X := \mathbb{P}^n$ given by

$$\sigma : \mathbb{G}_m \times \mathbb{P}^n \to \mathbb{P}^n \qquad\qquad (t, [x_0 : \cdots : x_n]) \mapsto [t^{-1}x_0 : tx_1 : \cdots : tx_n]$$

Obviously, the function $x_0 x_i$ for all $i \neq 0$ are invariants and we claim that these functions generate the ring of invariants.

Let $f \in k[x_0, \cdots, x_n]$, that is

$$f = \sum_{\underline{m}} a(\underline{m}) x_0^{m_0} \cdots x_n^{m_n}$$

where $\underline{m} = (m_0, \cdots, m_n)$. We have

$$t.f = \sum_{\underline{m}} a(\underline{m}) t^{m_1 + \cdots + m_n - m_0} x_0^{m_0} \cdots x_n^{m_n}$$

hence f in $\mathbb{G}_m$-invariant if, and only if, $a(\underline{m})$ vanish for all $\underline{m}$ such that $m_0 \neq \sum_{i=1}^n m_i$. And, when f is $\mathbb{G}_m$-invariant, we can write

$$f = \sum_{\underline{m}} a(\underline{m}) x_0^{m_0} \cdots x_n^{m_n} = \sum_{\underline{m}} a(\underline{m}) (x_0 x_1)^{m_1} \cdots (x_0 x_n)^{m_n}$$

Which implies that $k[x_0, \cdots, x_n]^{\mathbb{G}_m} \simeq k[x_0 x_1, \cdots, x_0 x_n] \simeq k[y_0, \cdots, y_{n-1}]$ taking the spectrum, we get $X//\mathbb{G}_m = \mathbb{P}^{n-1}$.

The ideal of invariant homogeneous polynomials of degree at least 1 is generated by $(x_0 x_1, \cdots, x_0 x_n)$ and the associated variety if $N = \{[x_0 : \cdots : x_n] | \, x_0 = 0 \text{ or } (x_1, \cdots, x_n) = 0\}$. Thus, the locus of the semi-stable points is $X^{ss} = \{[x_0 : \cdots : x_n] | \, x_0 \neq 0 \text{ or } (x_1, \cdots, x_n) \neq 0\} \simeq \mathbb{A}^n - \{0\}$. Moreover, every semi-stable point is stable as all orbits are closed in $\mathbb{A}^n - \{0\}$ and have zero dimensional stabilisers. We conclude that $X^{ss} = X^s = \mathbb{A}^n - \{0\} \to X//\mathcal{G}_m$ is a good quotient and since the preimage is a unique orbit, it is also an orbit space.

1.4.1 Linearisation of actions

A regular map from a projective variety X to $\mathbb{P}^n$ is equivalent to the data of a line bundle L and a set of its sections :

Let X be a variety, and let L be a line bundle on X. We say L is *base-point-free* if for every point $x \in X$, there is a global section of L which doesn't vanish. If this is true, then L determines a map to a projective space in the following way. The global sections of L are finite dimensional, so choose a basis (s_i). Then send a point $x \in X$ to

$$[s_1(x) : s_2(x) : ... : s_n(x)]$$

This leads us to the following definition :

Definition. Let X be a G-variety and $p : L \to X$ a line bundle on X. A *linearisation* of the action of G *with respect to* L is an action of G on L such that :

- for all $y \in L$, $g \in G$,

$$p(gy) = g.p(y)$$

- The map

$$L_x \to L_{g.x} \qquad\qquad y \mapsto gy$$

  is linear

Note that a linear action on L induces a linear action on $L^{\otimes r}$ and for any invariant section f of $L^{\otimes r}$, X_f is open and invariant.

Lemma 4. *Let L be a line bundle over X. Then the two assertions*

- *$\forall x \in X$, $\exists f$ a section of L^r (for some $r \in \mathbb{N}^*$) such that $f(x) \neq 0$ and X_f is affine,*

- *there exists a morphism $\psi : X \to \mathbb{P}^n$ auch that ψ maps isomorphically onto a quasi-projective variety in $\mathbb{P}^n$ ans $\psi^* H \simeq L^r$ for some r. Where H is a hyperplan bundle.*

are equivalent.
In this case, we called L ample.

We then define the notion of semi-stability and stability as in the previous case.

Definition. Let X be a projective G-variety with a line bundle L and a linearisation of G with respect to L. A point $x \in X$ is called

- *semi-stable* if, for some $r \in N^*$, there exists an invariant section f of L^r such that $f(x) \neq 0$ and $X_f = \{x \in X | \, f(x) \neq 0\}$ is affine,

- *stable* if it is semi-stable and $dim\ \mathcal{O}_x = dim\, G$ and the action of G on X_f is closed (for the f in the definition of semi-syability).

As before, we denote by $X^{ss}(L)$ and $X^s(L)$ respectively the set of semi-stable points and stable points.

Proposition 6. *Let X a projective G-variety in $\mathbb{P}^n$ and let L the line bundle obtained by restriction over X of the hyperplan bundle. Then any linear action on X induces a L-linear action and both definitions agrees :*

$$X^{ss}(L) = X^{ss}, \qquad X^s(L) = X^s$$

In some cases, given an action of a reductive group G on a projective variety X and a line bundle L, a linearisation of G with respect to L is unique.

Proposition 7. *Let L be a line bundle over X. Then an action of $SL_n(k)$ on X has at most one linearisation with respect to L.*

We have the following theorem, which generalize theorem 4.

Theorem 4. *Let X be a variety and L a line bundle over X. Then for any L-linear action of a reductive group G on X*

- *there exists a good quotient (Y, ϕ) of $X^{ss}(L)$ by G and Y is quasi-projective,*
- *there exists an open subset Y^s of Y such that $\phi^{-1}(Y^s) = X^s(L)$ and (Y, ϕ) is a geometric quotient of $X^s(L)$,*
- *for all $x_1, x_2 \in X^{ss}(L)$,*

$$\phi(x_1) = \phi(x_2) \Leftrightarrow \overline{\mathrm{O}_{x_1}} \cap \overline{\mathrm{O}_{x_2}} \cap X^{ss} \neq \varnothing$$

- *for x semi-stable point,*

$$x \in X^s(L) \Leftrightarrow dim\ \mathrm{O}_x = dim\, G,\ \text{and}\ \mathrm{O}_x\ \text{is closed in}\ X^{ss}(L)$$

Remark. The only differences between this theorem and theorem 4 are just that Y need not to be projective and that here we take in count the dependance on L.

1.4.2 More materials on linearisation (if we have time)

The definition of linearisation of an action σ of G on X can be reformulate by asking the diagram

$$\begin{array}{ccc} G \times L & \xrightarrow{\overline{\sigma}} & L \\ {\scriptstyle id\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow{\sigma} & X \end{array}$$

to be commutative and the zero section to be G-invariant.

We saw in the definition of linearisation of action σ with respect of a line bundle that the induced map on the fibers $L_x \to L_{g.x}$ is a linear isomorphism. Hence, we can also view the set of isomorphisms as an isomorphisms of the line bundle $\overline{\sigma}(g) : L \to g^*L$ and the conditions of action are translated to the following 1-cocycle condition

$$\overline{\sigma}(gg') = \overline{\sigma}(g') \circ g'^*\overline{\sigma}(g) : L \to g^*L \to g'^*(g^*L) = (gg')^*L$$

The collection of isomorphisms $\overline{\sigma}$ can be viewed as an isomorphism $\Phi : pr_2^*(L) \to \sigma^*L$, where $pr_2 : G \times X \to X$ is the natural projection. Moreover, we have

Lemma 5. *Let G be a connected affine algebraic group and X an algebraic G-variety. A line bundle L admits a G-linearisation if, and only if, there exists an isomorphism of line bundles $\Phi : pr_2^*(L) \to \sigma^*L$.*

Given two line bundle L and L' together with $\Phi : pr_2^*(L) \to \sigma^* L$ and $\Phi' : pr_2^*(L') \to \sigma^* L'$, one can construct their tensor product as the line bundle $L \otimes L'$ with the G-linearisation

$$\Phi \otimes \Phi' : pr_2^*(L \otimes L') = pr_2^*(L) \otimes pr_2^*(L') \to \sigma^*(L \otimes L') = \sigma^*(L) \otimes \sigma^*(L')$$

The zero element is the trivial line bundle $X \times \mathbb{A}^1$ with the trivial linearisation

$$\sigma \times id : G \times X \times \mathbb{A}^1 \to G \times X$$

and the inverse of (L, Φ) is $(L^{-1}, {}^t\Phi^{-1})$. We denote by $Pic^G(X)$ the abelian group defined by the set of line bundles L with isomorphisms $\Phi : pr_2^*(L) \to \sigma^* L$. We get a morphism

$$\alpha : Pic^G(X) \to Pic(X)$$

by forgetting the linearisation.

We obtain an exact sequence

$$0 \to Hom(G, k^*) \to Pic^G(X) \to Pic(X)^G \to H^2(G, k^*)$$

Proposition 8. *(Recall) Let L be a line bundle over X. Then an action of $SL_n(k)$ on X has at most one linearisation with respect to L.*

1.5 Criterion of stability

In this section, we will give a numerical criterion to stability due to Mumford. The main idea is to restrict the action of G to 1-parameter subgroups of G and work with this action.

Proposition 9. *Let X a G-variety and L a linearised line bundle. For a point $x \in X$, we denote by $\hat{x}$ a point in k^{n+1} lying over x. Then x is semi-stable if, and only if, $0 \notin \overline{\mathrm{O}_{\hat{x}}}$.*

Proof. If x is semi-stable then there exists f an invariant homogeneous polynomial of degree at least 1 such that $f(x) \neq 0$. Clearly $f(\hat{x}) \neq 0$ and $f(y)$ is equal to a non-zero constant for all $y \in \mathrm{O}(\hat{x})$ hence $0 \notin \mathrm{O}_{\hat{x}}$.

Conversely, if $0 \notin \overline{\mathrm{O}_{\hat{x}}}$ then by a previous lemma, one can find an invariant homogeneous polynomial such that $f(0) = 0$ and $f(y) = 1$ for all $y \in \mathrm{O}(\hat{x})$. But f has constant term equal 0 so, there exists some homogeneous part of f of degree at least 1 such that it is not 0 at $\hat{x}$. □

Hence, one can detect some unstable points by checking if $0 \in \overline{H.\hat{x}}$ for some subgroup H os G. If we consider H to be the image of a 1-parameter subgroup $\lambda : \mathbb{G}_m \to G$ one has, in appropriate coordinates

$$\lambda(t).\hat{x} = (t^{m_0} x_0, \cdots, t^{m_n} x_n)$$

Suppose that all m_i for which $x_i \neq 0$ are stricly positive. Then the map

$$\lambda_{\hat{x}} : \mathbb{A}^1 - \{0\} \to \mathbb{A}^{n+1}, \qquad t \to \lambda(t).\hat{x}$$

can be extended to $\mathbb{A}^1$ by sending the zero to the origin of $\mathbb{A}^{n+1}$. In this case, this is clear that $0 \in \overline{\mathrm{O}_{\hat{x}}}$ and x is unstable.

Remark. More precisely, since we consider the case where X is projective, the map

$$\lambda_x : \mathbb{A}^1 - \{0\} \to X, \qquad\qquad t \mapsto \lambda(t).x$$

can be extended to $\overline{\lambda_x} : \mathbb{P}^1 \to X$ and we set $\lim_{t\to 0} \lambda(t).x := \overline{\lambda_x}(0)$ and in the same way for ∞.

If all m_i for which $x_i \neq 0$ are strictly negative then by $\lambda^{-1}(t) := \lambda(t^{-1})$ we reach to the same conclusion.

If we set

$$\mu(x, \lambda) := min_i \{m_i | \, x_i \neq 0\}$$

then we can reformulate this remark by saying that x is unstable if there exists a 1-parameter subgroup $\lambda : \mathbb{G}_m \to G$ such that $\mu(x, \lambda) > 0$. Hence, we get

$$\text{if } x \text{ is semi-stable then } \mu(x, \lambda) \leqslant 0, \; \forall \lambda : \mathbb{G}_m \to G$$

Assume now that $\mu(x, \lambda) = 0$ for some λ and x is stable. Let $y = (y_0, y_1, \cdots, y_n)$, where $y_i = x_i$ if $m_i = 0$ and $x_i \neq 0$ and $y_j = 0$ otherwise. Then, by taking the limit when t tends to 0, we see that $y \in \overline{\mathrm{O}_x}$. If x was a stable point, then its orbit is closed and y is in O_x but y is fixe by $\lambda(\mathbb{G}_m)$ so it is not stable. This contradict the fact that x is stable. Hence, we get another characterization

$$\text{if } x \text{ is table then } \mu(x, \lambda) < 0 \; \forall \lambda : \mathbb{G}_m \to G$$

And actually, the following theorem says that the converse is true.

Theorem 5. *Let G be a reductive group acting on a projective algebraic variety X and L be an ample G-linearised line bundle over X. Then a point $x \in X$ is*

- *stable if, and only, if $\mu(x, \lambda) < 0, \; \forall \lambda : \mathbb{G}_m \to G$,*
- *semi-stable if, and only, if $\mu(x, \lambda) \leqslant 0, \; \forall \lambda : \mathbb{G}_m \to G$,*

Proposition 10. *Let $SL_n(\mathbb{C})$ act linearly on a projective variety X. A point x is stable (resp. semi-stable) if and only if, $\mu(g.x, \lambda) < 0$ (resp. $\leqslant$) for every $g \in SL_n(\mathbb{C})$ and every 1-par. subgroup λ.*

Example. Consider the moduli problem given by plane cubics curve in $\mathbb{CP}^2$ up to the action of $GL_3(\mathbb{C}) = \mathbb{C}^* \times SL_3(\mathbb{C})$.

A plane cubic is defined by a non zero polynomial up to scalar multiplication.

$$\begin{aligned} f = a_{3,0}X_1^3 + a_{2,1}X_1^2X_2 + a_{1,2}X_1X_2^2 + a_{0,3}X_2^3 + \\ a_{2,0}X_0X_1^2X_2 + a_{1,1}X_0X_1X_2 + a_{0,2}X_0X_2^2 + \\ a_{1,0}X_0^2X_1 + a_{0,1}X_0^2X_2 + a_{0,0}X_0^3 \end{aligned}$$

$$\text{i.e. } f = \sum_{i+j\leqslant 3} a_{i,j} X_0^{3-i-j} X_1^{r_1} X_2^{r_2}$$

To avoid a complete lecture, we take a characterization on the coefficients to get singularities and up to the action of $SL_3(\mathbb{C})$ we can look at singularities at $(1, 0, 0)$.

- $(1,0,0)$ is an ordinary double point if, and only if,

$$a_{0,0} = a_{1,0} = a_{0,1} = 0$$

- $(1,0,0)$ is a non ordinary double point if, and only if,

$$a_{0,0} = a_{1,0} = a_{0,1} = 0 \text{ and } a_{2,0}a_{0,2} = \frac{1}{4}a_{1,1}^2$$

- $(1,0,0)$ is a triple point if, and only if,

$$a_{0,0} = a_{1,0} = a_{0,1} = a_{2,0} = a_{1,1} = a_{0,2} = 0$$

Proposition 11. *The set of stable cubics correspond to non-singular one. The set of semi-stable cubics are the singular one without non ordinary double point or triple point.*

Proof. We the previous proposition, we can only look at $\mu(f,\lambda)$ for a particular λ to characterize non-semi-stability or non-stability. For example, one can take

$$\lambda : t \to diag(t^{r_0}, t^{r_1}, t^{r_2})$$

where $\sum r_i = 0$ with $r_0 \leqslant r_1 \leqslant r_2$. And for such λ,

$$\lambda(t).f(X_0, X_1, X_2) = f(t^{r_0}X_0, t^{r_1}X_1, t^{r_2}X_2) = \sum_{i,j=0}^{3} t^{(3-i-i)r_0+ir_1+jr_2} a_{i,j} X_0^{3-i-j} X_1^{r_1} X_2^{r_2}$$

and

$$\mu(f,\lambda) = min_{i+j\leqslant 3}\left\{(3-i-j)r_0 + r_1 i + r_2 j|\, a_{ij} \neq 0\right\}$$

Denote by $E_{i,j} := (3-i-j)r_0 + r_1 i + r_2 j$. By assumption on the r_i, we have that

$$E_{0,0} \leqslant E_{1,0} \leqslant E_{0,1} \leqslant E_{2,0} \leqslant E_{1,1} = 0$$

We know decompose into 2 parts corresponding to unstability and semi-stability.

- We see that $\mu(f,\lambda) > 0$ (equivalent to f is unstable) implies that $E_{0,0} \leqslant E_{1,0} \leqslant E_{0,1} \leqslant E_{2,0} \leqslant E_{1,1} = 0$ and does not appears in the minimum and hence the corresponding $a_{i,j}$ vanish. Conversely, if f satisfies

$$a_{0,0} = a_{1,0} = a_{0,1} = a_{1,1} = a_{2,0} = 0$$

we can take $r_0 = -3$, $r_1 = 1$ and $r_2 = 2$ and check that $\mu(f,\lambda) > 0$. Thus, if f has a triple or f has a double point with a unique tangent, it is unstable. (for the case where $a_{0,2} = 0$, we take $r_0 \leqslant r_2 \leqslant r_1$ to conclude for f has a non ordinary double point then it is unstable).

- If f has an ordinary double point (always assume to be at $(1,0,0)$), then we can check that $\mu(f,\lambda) = r_0 + 2r_1 \leqslant r_0 + r_1 + r_2 = 0$ (because either $a_{1,1}$ or $a_{2,0}$ are not 0) by assumption on r_i. Moreover for suitable r_i one can have $\mu(f,\lambda) = 0$ for $a_{0,0} = a_{1,0} = a_{0,1} = 0$. So a singular cubic with ordinary double point is semi-stable.

  Finally, for f non-singular we have $\mu(f,\lambda) \leqslant 3r_0 < 0$ and r_0 have to be negative by assumptions so that f is stable. For the final part of this proof, see [2].

□

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